Last time: showed that

$$(\gamma^\mu p_\mu - m)\psi(p) = 0$$

is Lorentz invariant

$\rightarrow$ can examine it in frame

$$p^\mu = (m, \vec{0}) : \quad (\gamma^0 - 1)\psi = 0 \qquad (1)$$

as $(\gamma^0 - 1)^2 = (\gamma^0)^2 - 2\gamma^0 + 1 = -2(\gamma^0 - 1)$

$\rightarrow$ up to normalization $(\gamma^0 - 1)$ is projection operator (recall: $P^2 = P$)

Using $\gamma_0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$

we see

$$(1) \rightarrow \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix} \psi = 0$$

thus 2 out of 4 components of ψ vanish! Makes sense as electron has 2 degrees of freedom (and not 4)

Similarly, the Klein-Gordon eq.

$$(\partial^2 + m^2)\varphi(x) = 0$$

just projects out Fourier modes not satisfying

$$k^2 = m^2$$

→ equations of motion in relativistic physics just project out unphysical components

<u>Convention:</u> $\not{a} := \gamma^\mu a_\mu$

→ Dirac eq.: $(i\not{\partial} - m)\psi = 0$

<u>Cousins of the gamma matrices</u>

have 16 linearly independent 4×4 matrices

- $\mathbb{1}, \gamma^0, \gamma^1, \gamma^2, \gamma^3 \longrightarrow 5$
- $\gamma^5 := i\gamma^0\gamma^1\gamma^2\gamma^3 \longrightarrow 1$

$$= \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

$$\{\gamma^5, \gamma^\mu\} = 0$$

- $\gamma^\mu\gamma^5 = \pm i \gamma^\mu\gamma^\nu\gamma^\lambda \longrightarrow 4$

  with μ, ν, λ different

- $\sigma^{\mu\nu} := \frac{i}{2}[\gamma^\mu, \gamma^\nu] \longrightarrow 4 \cdot 3/2 = 6$

  $(J^{\mu\nu} = \frac{1}{2}\sigma^{\mu\nu})$

Altogether we have found

$$5 + 1 + 4 + 6 = 16$$

matrices: $\{ \mathbb{1}, \gamma^\mu, \sigma^{\mu\nu}, \gamma^\mu\gamma^5, \gamma^5 \}$

any 4×4 matrix can be written as a linear combination of these!

We have

$$\sigma^{0i} = i \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}, \quad \sigma^{ij} = \varepsilon^{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$$

Dirac bilinears

Note that γ^0 is hermician while γ^i is antihermitian

$$\longrightarrow \quad (\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0$$

$\longrightarrow$ $\psi^\dagger \gamma^\mu \psi$ is not hermitian

but $\bar{\psi} \gamma^\mu \psi$ with $\bar{\psi} := \psi^\dagger \gamma^0$ is!

also $(\sigma^{\mu\nu})^\dagger = \gamma^0 \sigma^{\mu\nu} \gamma^0$

Hence

$$U(\Lambda)^\dagger = \gamma^0 e^{\frac{i}{2}\omega_{\mu\nu} J^{\mu\nu}} \gamma^0$$

so

$$\bar{\psi}'(x') = \psi(x)^\dagger U(\Lambda)^\dagger \gamma^0$$
$$= \bar{\psi}(x) e^{+\frac{i}{2}\omega_{\mu\nu} J^{\mu\nu}}$$

and we get

$$\bar{\psi}'(x')\psi'(x') = \bar{\psi}(x) e^{+\frac{i}{2}\omega_{\mu\nu} J^{\mu\nu}} e^{-\frac{i}{2}\omega_{\mu\nu} J^{\mu\nu}} \psi(x)$$
$$= \bar{\psi}(x)\psi(x)$$

$\rightarrow$ $\bar{\psi}(x)\psi(x)$ transforms as Lorentz scalar (and <u>not</u> $\psi^\dagger\psi$)

There are 16 Dirac bilinears we can form: $\bar{\psi}\Gamma\psi$

where $\Gamma \in \text{span}\{\mathbb{1}, \gamma^\mu, \gamma^5, \gamma^5\gamma^\mu, \sigma^{\mu\nu}\}$

one finds

$\bar{\psi}\gamma^\mu\psi$ is Lorentz vector

$\bar{\psi}\sigma^{\mu\nu}\psi$ is Lorentz scalar

$\bar{\psi}\gamma^5\psi$ is pseudoscalar

$\bar{\psi}\gamma^\mu\gamma^5\psi$ is pseudovector

The term "pseudo-scalar" or "pseudo-vector" means that the object transforms as a scalar/vector under continuous Lorentz transformations but picks up a sign under "parity"

Parity:

Important symmetry in physics: reflection in a mirror or "parity"

$$x^\mu \longmapsto x'^\mu = (x^0, -\vec{x})$$

A the level of the Dirac eq.:

$$\gamma^0 (i\gamma^\mu \partial_\mu - m)\psi(x) = 0$$

$$= (i\gamma^\mu \partial'_\mu - m)\gamma^0 \psi(x) \quad \text{where} \quad \partial'_\mu = \frac{\partial}{\partial x'^\mu}$$

$\longrightarrow$ $\psi'(x') := \eta\gamma^0\psi(x)$ satisfies parity-reflected Dirac-eq (η arbitrary phase, set $\eta = 1$)

Note: $\bar{\psi}'(x')\psi'(x') = \bar{\psi}(x)\psi(x)$

but $\bar{\psi}'(x')\gamma^5\psi'(x') = \bar{\psi}(x)\gamma^0\gamma^5\gamma^0\psi(x)$

$$= -\bar{\psi}(x)\gamma^5\psi(x)$$

The Dirac Lagrangian

set $\mathcal{L} = \bar{\psi}(i\cancel{\partial} - m)\psi$

$$\rightarrow \quad \partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \psi} - \frac{\delta \mathcal{L}}{\delta \psi}$$

$$= \partial_\mu (i \bar{\psi} \gamma^\mu) + m\bar{\psi} = 0$$

$\rightarrow$ gives Dirac equation upon hermitian conjugation and multiplication by γ^0

Another equation is obtained by varying $\bar{\psi}$:

$$\partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \bar{\psi}} - \frac{\delta \mathcal{L}}{\delta \bar{\psi}} = 0$$

$\rightarrow$ Dirac eq.

Slow and fast electrons

Dirac eq. in momentum space:

$$(\cancel{p} - m)\psi(p) = 0 \qquad (2)$$

$\rightarrow$ simple matrix eq.

Using $\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}$, the eq. in rest frame (eq. (1)), tells us $\begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix} \begin{pmatrix} \phi \\ \chi \end{pmatrix} = \begin{pmatrix} 0 \\ \chi \end{pmatrix} = 0$

$\Rightarrow \chi = 0$

Thus, for a slowly moving electron we expect: $\chi(p) \ll \phi(p)$

On the other hand, for $p^2 \gg m^2$,

$$(2) \longrightarrow \quad \not{p}\,\psi = 0 \qquad \Big| \cdot \gamma_5$$

$$\Longleftrightarrow \gamma^5 \not{p}\,\psi = -\not{p}\,\gamma^5 \psi = 0$$

Since $\gamma^5 = \mathbb{1}$, we can form projection operators

$$P_L := \frac{1}{2}(1-\gamma^5) \quad , \quad P_R := \frac{1}{2}(1+\gamma^5)$$

satisfying $P_L^2 = P_L$, $P_R^2 = P_R$, $P_L P_R = 0$

$\rightarrow$ introduce $\psi_L = \frac{1}{2}(1-\gamma^5)\psi$,

$$\psi_R = \frac{1}{2}(1+\gamma^5)\psi$$

note: $\gamma^5 \psi_L = -\psi_L$, $\gamma^5 \psi_R = \psi_R$

$\longrightarrow$ two degrees of freedom of electron called "left-handed" or "right-handed"

$\longrightarrow$ introduce new basis for fast-moving electrons, "Weyl basis":

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \longrightarrow \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}$$

and we have $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$

ψ_L and ψ_R are known as "Weyl spinors"

Defining $\sigma^\mu = (\mathbb{1}, \vec{\sigma})$, $\bar{\sigma}^\mu = (\mathbb{1}, -\vec{\sigma})$, we can write

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$

in Weyl basis.

<u>Chirality or handedness</u>

We can always decompose a Dirac field $\psi(x)$ as

$$\psi(x) = \psi_L(x) + \psi_R(x)$$
$$= \frac{1}{2}(1-\gamma_5)\psi(x) + \frac{1}{2}(1+\gamma_5)\psi(x)$$

$\longrightarrow$ Lagrangian can be written as

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi = \bar{\psi}_L i\partial\!\!\!/ \psi_L + \bar{\psi}_R i\partial\!\!\!/ \psi_R$$
$$- m(\bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L)$$

- The transformation $\psi \mapsto e^{i\theta}\psi$ leaves $\mathcal{L}$ invariant

  $\longrightarrow$ Noether current $j^\mu = \bar{\psi}\gamma^\mu\psi$

  check: $\partial_\mu j^\mu = (\partial_\mu\bar{\psi})\gamma^\mu\psi + \bar{\psi}\gamma^\mu\partial_\mu\psi$
  $= (im\bar{\psi})\psi + \bar{\psi}(-im\psi) = 0$

- If $m=0$, $\mathcal{L}$ has an additional symmetry, called "chiral symmetry":

  $$\psi \longmapsto e^{i\phi\gamma^5}\psi$$

  or $\psi_L \mapsto e^{-i\phi}\psi_L$, $\psi_R \mapsto e^{i\phi}\psi_R$

  $\longrightarrow$ Noether current: $j^{5\mu} = \bar{\psi}\gamma^\mu\gamma^5\psi$

  check: $\partial_\mu j^{5\mu} = 2im\bar{\psi}\gamma^5\psi$